

Variations on Birkhoff's theorem

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April 12, 2013

Abstract

The relation between the expanding universe and local vacuum solutions, such as that for the Solar System, is crucially mediated by Birkhoff's theorem. Here we consider how that relation works, and give generalizations of Birkhoff's theorem when there are geometric and matter and perturbations. The issue of to what degree dark matter might influence the solar system emerges as a significant question.

1 Birkhoff's theorem

Birkhoff's theorem in General Relativity [3, 26, 13], actually first discovered by Jebsen [28] (see [29]), states that any local spherically symmetric solution of the vacuum Einstein field equations

$$R_{\mu\nu} = 0 \tag{1}$$

must admit an extra Killing vector, and so be part of the Schwarzschild vacuum metric:

$$ds^2 = - \left(1 - \frac{2m}{r}\right) + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \tag{2}$$

where m is a constant representing the mass of the central object (if there is no central mass so that $m = 0$, this is just flat spacetime). If the Killing vector is timelike, the spacetime is locally static, and is asymptotically flat if it extends far enough; the local solution is part of the exterior region ($r > 2m$) of (2). If the Killing vector is spacelike, the spacetime is locally spatially homogeneous: it is part of the interior black hole region ($r < 2m$) of (2), and runs into a singularity if it extends far enough.

The basic idea is that a spherically spherical object should produce a spherically spherical gravitational field; another mass elsewhere, or anisotropic boundary conditions, would

disturb the spherical symmetry. So the solution represents an isolated spherical object; the theorem then says the exterior spacetime is static. It should be emphasized that this is a local result in the sense that it only depends on the field in some open domain U being vacuum and spherically symmetric in that domain; then that domain is either static or spatially homogeneous. The proof is local (see section 3.2 below), and so does not directly use boundary conditions at infinity, although those conditions will indirectly affect the outcome because they do influence whether the region U is spherically symmetric or not.

A consequence is that no radial changes in a spherical star (whether it is expanding, pulsating, collapsing, or whatever) affect its external gravitational field: all spherically symmetric vacuum gravitational fields are indistinguishable for $r > R_s > 2m$ where $R_s(t)$ is the coordinate value at the surface of the star. This means a spherically pulsating star cannot emit gravitational waves, nor gravitationally radiate away its mass.

In summary, the Schwarzschild solution is the unique spherically symmetric solution of the vacuum Einstein field equations (1): a spherically symmetric gravitational field in empty space outside a star must be static, with a metric given by (2) for $r > 2m$. It represents the spacetime of the Solar System, and all other spherically symmetric star systems, to very good approximation, and so is key in much astrophysics and astronomy.

If the cosmological constant is non-zero, the result essentially remains true. The Kottler metric [37], found independently by Kottler [30] and Weyl [44], is the unique spherically symmetric solution of the field equations with a cosmological constant $\Lambda \neq 0$:

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad (3)$$

has solution

$$ds^2 = - \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} \right) + \frac{dr^2}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (4)$$

It is also known as the Schwarzschild–de Sitter metric for $\Lambda > 0$ and the Schwarzschild–anti-de Sitter metric for $\Lambda < 0$. If $\Lambda = 0$ it is the Schwarzschild solution; if $m = 0$, it is just the de Sitter or anti-de Sitter metric. The global structure is complex, depending on the relation between m and Λ (see [32]).

Birkhoff's theorem can be generalized to some matter fields: any spherically symmetric solution of the Einstein–Maxwell field equations must be static in the exterior domain, with a Reissner–Nordström metric ([13], section 18.1).¹ However it does not hold for matter such as baryons or a perfect fluid: their spherically symmetric solutions have a dynamic behavior [4]. But systems such as the solar system are not dominated by a perfect fluid or any other continuous matter: they are basically empty space with isolated bodies embedded in the vacuum. To a large extent the same is true of galaxies, mainly made up of isolated stars, and clusters of galaxies, mainly made up of isolated galaxies.

¹This is of academic interest only, as charged stars do not exist in reality. If they did, astronomy would be governed by electric forces rather than gravity, because gravity is such a weak force.

With a caveat about dark matter (see below), most of the universe is empty space. In any case even if dark matter is present, all these systems are presently either static, or at least stationary, to a very good approximation.

This raises a dilemma: if almost all local stellar systems are static, how can they be put together to give an expanding universe? Putting static domains together to make an expanding spacetime is non-trivial. That is the topic of Section 2.

Birkhoff's theorem underlies the crucial importance in astrophysics of the Schwarzschild solution, as it means that the exterior metric of any exactly spherical star must be given by the Schwarzschild metric and this also underlies the uniqueness results for non-rotating black holes. However it is an exact theorem that is only valid for exact spherical symmetry; but no real star system is exactly spherically symmetric (for example if they have planets). So a key question is whether the result is approximately true for approximately spherically symmetric vacuum solutions. In Section 3, we prove an "almost Birkhoff theorem" for this case [23] where spherical symmetry is not exact. Furthermore the Solar System is not exactly empty: it is pervaded by very low density material. In Section 4 we prove a second "almost Birkhoff theorem" for that case too [24]. So the results carry over to astrophysically realistic situations (such as the Solar System), which are not empty and where spherical symmetry is not exact.

The key issue that becomes clear through this study is the question of whether or not dark matter pervading systems such as the solar system links local systems to the cosmological expansion. If it does, then in principle one can measure the Hubble constant by solar system measurements.

This paper is an extension of a talk at the Spanish Relativity Society meeting ERE2012 at Guimarães by one of us (GFRE).

2 Birkhoff's theorem and the expanding universe

Birkhoff's theorem plays a key role as regards the relation of the expanding universe and local vacuum solutions. The issue is, does the Hubble expansion affect local physical systems such as the solar system? Birkhoff's theorem says no, it does not if the system is empty except for a spherically symmetric central object, so that spacetime is locally spherically symmetric. The solution is then locally static and hence completely decoupled from the global expansion. Therefore you cannot measure the Hubble constant in the solar system: it has decoupled from the universal expansion.

But this applies almost everywhere. The universe is largely comprised of empty space, with isolated compact objects scattered in the void. Galaxy clusters are mainly empty space, as are galaxies themselves, with immense empty regions separating the isolated stars that are the bulk of matter in galaxies. Indeed most of the baryonic matter in galaxies is concentrated in isolated stars surrounded by empty space. Most of the universe is locally static (or perhaps stationary, if the matter is rotating). Now it is true that this pictures

is complicated by the issue of how clustered dark matter is, given that dark matter is the bulk of matter on a cosmological scales; so maybe it is not empty locally after all. We will return to this issue later: for the moment we concentrate on cases where this is a good approximation.

2.1 Vacuum almost everywhere

Consider a universe that is locally made of spherically symmetric vacuum regions (such as the Solar System), which are static, because of Birkhoff's theorem. They need to somehow be joined together to give a globally expanding, approximately spatially homogeneous spacetime. Thus locally rigid domains are joined to give a globally expanding spacetime. How is it done? We will look successively at perturbed FLRW models (Section 2.2); Swiss Cheese models (Section 2.3); Lindquist-Wheeler models (Section 2.4); and the exact two-mass version of those models (Section 2.5).

2.2 Perturbed FLRW models

The standard models for structure formation in cosmology are perturbed Friedmann-Lemaître-Robertson-Walker (FLRW) models with inhomogeneities imbedded, as represented in a linearised solution of the Einstein Equations [20, 35]. However they are fluid filled everywhere, and so do not represent the situation sketched above. The FLRW regions expand and carry the inhomogeneities representing structure such as galaxies with them; these models do not represent local empty static domains, such as the Solar System. They may perhaps be able to locally represent virialised regions that have in effect opted out of the cosmic expansion, but whether this can be done in a *single* coordinate system that represents *all* such locally static systems is not clear.

One may possibly be able to do it for a region where the Hubble velocities are small near the origin of one coordinate system, if one uses a static non-comoving frame for that domain (see equation (2.1) in [1] for steps in this direction); but this does not give the desired representation for local static systems anywhere else. Thus one can perhaps do this for a particle Q at the origin of coordinates $x^i(Q)$, correctly representing its local domain $U(Q)$ as exactly static; or else for a different distant particle P , correctly representing its local domain $U(P)$ as static, using a different set of coordinates $x^i(P)$ centered in P . But these are different coordinate systems. The problem is you can't do it simultaneously for both regions using one single coordinate system, nor *a fortiori* for 10^{11} separate static domains at all local locations across the entire visible universe that abut each other with no intervening fluid filled domains. But that is what is needed to represent the locally static nature of spacetime everywhere.

Overall these models do not show how local static domains fit together to give expansion, because they rely on an all-pervading cosmic fluid model for their dynamics, and that fluid everywhere embodies the expansion of the universe. It is perhaps not impossible to show how one can get local static domains in such models, but remember we have to do that *everywhere*: on the view put here, the universe is globally made up of locally static vacuum domains with isolated stars imbedded in them. That is what those

coordinates probably cannot represent. These models rather represent the case where relative velocities decrease towards zero at small scales, but are never actually zero for any finite distance, because there is an expanding cosmic fluid everywhere. It is likely one could measure the Hubble constant in the Solar system in such models, if we take them as extending down to that scale.

2.3 Swiss cheese models

The Einstein-Strauss “Swiss Cheese” models [15, 16] do indeed properly represent local static domains such as the solar system. They are based on a FLRW background model with vacuum regions (“vacuoles”) cut out, and carefully matched with the fluid solutions at the boundaries between the empty regions and the cosmological fluid. The vacuum domains can have static star models imbedded at the centre, to give non-singular vacuoles imbedded in a FLRW model. One can do this for vast numbers of such vacuoles, considered as representing spacetime around either stars or galaxies, and can even do it in a hierarchical way that includes both [38, 36].

There is no problem with expansion in this case: the connected FLRW regions surrounding the vacuoles do indeed expand, and carry the static vacuoles with them. One cannot measure H_0 in the Solar System in these models, as the vacuoles are static. However these models do not solve the issue posed here, because the background FLRW model in them embodies the expansion and carries the vacuoles apart from each other. We want to model the case where there are no such fluid-filled domains occupying the space between stars or galaxies.

Thus there is no problem with global expansion if globally interconnected fluid domains are allowed to surround the static vacua as in the Swiss Cheese models, but we want to know what happens if there is vacuum everywhere except in interiors of stars.

2.4 Lindquist-Wheeler

In an innovative paper, Lindquist and Wheeler [34] accurately modelled the situation considered here by considering a regular lattice of Schwarzschild vacuum cells joined together to give an expanding solution. There are no fluid filled regions in these models; it is entirely made of static domains with imbedded stars, as envisaged above. They average to a FLRW spacetime with closed spatial sections when considered on large scales, and an effective Friedmann equation results from the junction conditions at the boundaries between the static domains.

Thus unlike the previous two cases, there is NO background FLRW spacetime in this case, and no connected fluid that expands and carries the stars and galaxies with them. The FLRW model emerges as a large scale averaged approximation [34]. The Locally Static vacuum domains recede from each other because of the boundary conditions at the joins between them. Ferreira and Clifton have recently extended these models in interesting ways [9, 10], using perturbation methods.

These are very good models in terms of tackling the issue posed here, but they are not exact solutions: the junction conditions in [34] are not very clear, because the lattice symmetry means the spherical symmetry of the vacuoles is not exact. The Schwarzschild solutions in each vacuole are an approximation to the real geometry because of this anisotropy (although one can get exact solutions for the corresponding initial value problem [11]). Clifton and Ferreira give helpful perturbative solutions of this kind [9, 10]; but the issue we want to look at is, are there any exact solutions of this nature?

2.5 Two mass solution

One can indeed find exact such solutions in a very simple case: a 2-mass solution of this kind, with compact spatial sections because of a positive Λ term [41]. In order to understand how locally static configurations around gravitationally bound bodies can be embedded in an expanding universe, that paper investigate the general relativity solutions describing a space-time whose spatial sections have the topology of a 3-sphere with two identical masses at the poles. Introduction of a cosmological constant allows closed space sections.

One envisages two massive objects M_1, M_2 of equal mass M , each embedded in a vacuum spherically symmetric domain U_1, U_2 to give a spherical stellar model surrounded by empty space (no horizon occurs, because their surfaces R_1, R_2 lie outside their Schwarzschild radii). Each vacuum domain is a segment of a Kottler solution (4) with 2-sphere surface area at coordinate value r given by $A = 4\pi r^2$. The cosmological constant $\Lambda > 0$ is chosen large enough so the area A reaches a maximum at a value $A_* = 4\pi r_*^2$, and U_1, U_2 are joined back to back at this value $r_*(\Lambda, M)$, to give closed space sections. Thus we form a vacuum spacetime with compact spatial sections and identical antipodal stellar masses; it is spherically symmetric about each of the two masses.

The technical issue is matching the solutions at r_* using the Israel junction conditions: the first and second fundamental forms must be continuous so that no surface layer occurs. Because r_* is a constant (it can depend only on M and Λ , which are both constant) it seems at first glance that this model cannot expand: only solutions with a fixed size are possible. One should obtain the analogue of the Einstein Static universe: a two mass exact solution that is static, and vacuum everywhere except at the two antipodal stars.

However closer inspection shows that such static solutions are not possible: one can't simultaneously match both the spatial and time components of the second fundamental form. Israel junction conditions imply that two spherically symmetric static regions around the masses cannot be glued together in the desired way. The resolution [41] is that one must match across a null horizon, like the way de Sitter universe has local static domains matched to expanding domains across null horizons [39, 26]. The Penrose diagram for this situation is shown in Figure 1, and an embedding diagram in Figure 2.

The de Sitter static frame covers only part of the de Sitter hyperboloid; taking spatial sections $\{t = \text{constant}\}$ in the imbedding diagram, those static sections are separated from each other by expanding domains across 2-spheres that are the null horizons (top

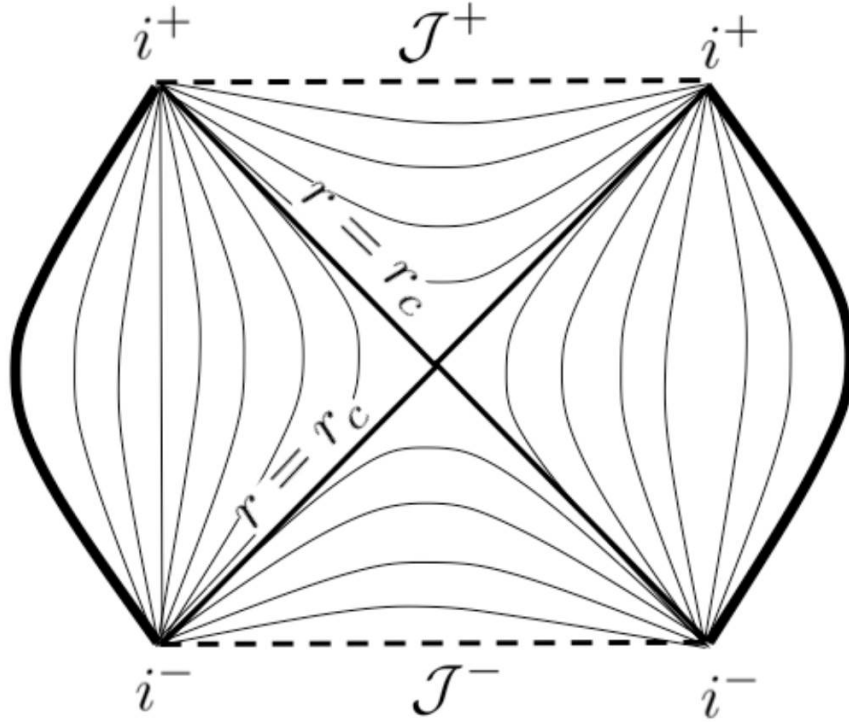


Figure 1: *The Penrose diagram for the two-mass Kottler solution [41]. There are no event horizons surrounding the antipodal origins (these are filled in by interior star solutions).*

inset in Figure 2); those spatial sections decrease from infinity, reach a time-symmetric minimum (bottom inset in Figure 2), and re-expand. The static model we tried to find is just the situation at that minimum radius: the matching we sought is possible at that instant (and only then), because the second fundamental form instantaneously vanishes there. Thus study of the extension of the Kottler space-time shows that there exists a non-static solution consisting of two static regions surrounding the masses, that are each matched to a Kantowski-Sachs expanding region at a null horizon. It is the expanding vacuum domains that interpose between the static domains that allows the universe to expand.

There must be a set of coordinates for this solution corresponding to the de Sitter $k = +1$ frame with expansion parameter $a(t) = \cosh Ht$. **To be done:** find a global perturbed FLRW coordinate system for this model, like a perturbed de Sitter hyperboloid, for the case when the mass is very small; and see how to extend it to cases where the mass is large.

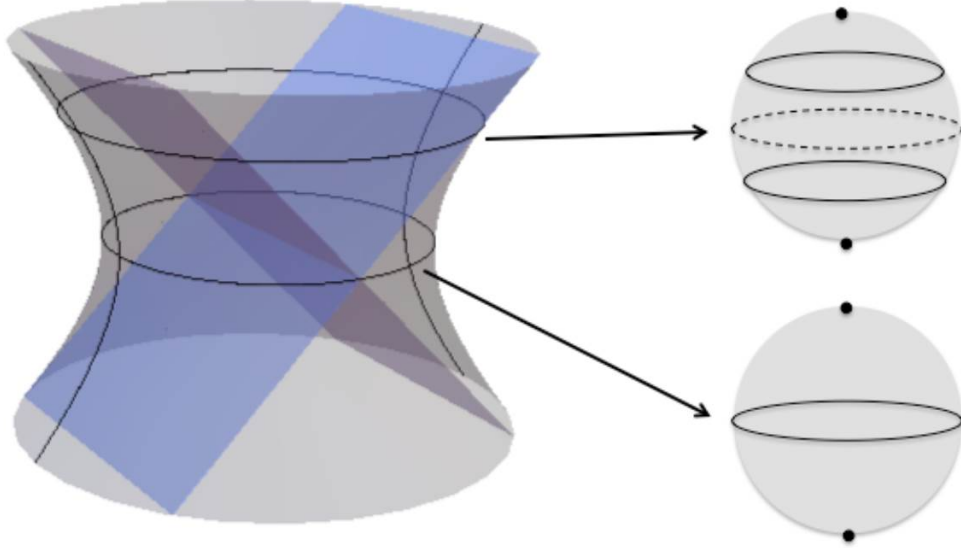


Figure 2: *Embedding of the 2-mass solution in 5-d spacetime: De Sitter hyperboloid, with two antipodal masses imbedded [41]. Inset: the spatial sections $\{t = \text{const.}\}$ at a generic time (above) and at the time symmetric throat (below).*

2.6 Dark matter and local systems

The two mass solution shows how static vacuum domains can be joined together to form an expanding universe: you glue them together across expanding vacuum regions.

Now if that the vacuum dominated model proposed above is all wrong and there are no such vacuum domains (e.g. in the solar system) because of dark matter, that analysis will not apply: the universe can be built up out of solar-system like domains because they are pervaded by dark matter which takes part in the cosmic expansion, hence they are not exactly static (as must be the case if they are vacuum filled domains, because of Birkhoff's theorem). And in that case you will indeed be able to measure the Hubble constant in the solar system. The same issue arises as regards the galaxy: if its internal dynamics is dominated by dark matter, the question arises as to whether that dark matter is taking part in the Hubble expansion or not. Thus there are two options:

The locally static case: The local domains may still be essentially static, and something like the two mass model must apply, even when there is not a vacuum. Indeed one may surmise this is indeed the case: galaxies are not in violent internal radial motion, and the Hubble parameter does not enter analyses of galactic dynamics [2]. In that case the essential ideas of the two mass solution will still apply, even though the details will be different: locally static regions can expand because they are separated from each other by expanding domains.

The locally non static case: If this is not true, and a very low density of dark matter pervading the solar system is indeed taking part in the cosmic expansion, then these non-static domains can be stitched together to give an expanding universe; and we can least in principle measure the Hubble constant in the solar system, opening up a new avenue of investigation in cosmology.

3 Almost Birkhoff: perturbations

The core content of Birkhoff's theorem [3] is that any spherically symmetric solution of the vacuum field equations has an extra symmetry: it must be either locally static or spatially homogeneous. This underlies the crucial importance in astrophysics of the Schwarzschild solution, as it means that the exterior metric of any exactly spherical star must be given by the Schwarzschild metric and it also underlies the uniqueness results for non-rotating black holes. However it is an exact theorem that is only valid for exact spherical symmetry; but no real star is exactly spherically symmetric. So a key question is whether the result is approximately true for approximately spherically symmetric vacuum solutions. In this section, we summarise the proof of an “almost Birkhoff theorem” for that case [23], so those results carry over to astrophysically realistic situations (such as the Solar System) where spherical symmetry is not exact.

There are of course many papers discussing perturbations of the Schwarzschild solution, but none appear to focus on this specific issue. The rigidity embodied in this property of the Einstein Field Equations is specific to vacuum General Relativity solutions, or solutions with a trace-free matter tensor. One should also note that the result is local: both Birkhoff's theorem, and our generalization of it, are independent of boundary conditions at infinity: they hold in local neighborhoods U .

3.1 1+1+2 covariant splitting of spacetime

We prove the result by using the 1+1+2 covariant formalism [8, 7], which developed from the 1+3 covariant formalism.

In the 1+3 covariant formalism [14, 18], first we define a preferred timelike congruence with a timelike unit tangent vector u^a ($u^a u_a = -1$). Then the tangent spaces to spacetime are split in the form $R \otimes V$ where R denotes the timeline along u^a and V is the tangent 3-space perpendicular to u^a .

In the (1+1+2) approach we further split the tangent 3-space V , by introducing a spacelike unit vector e^a orthogonal to u^a so that $e_a u^a = 0$, $e_a e^a = 1$. Then the *projection tensor* $N_a^b \equiv g_a^b + u_a u^b - e_a e^b$ projects vectors onto the tangent 2-surfaces orthogonal to e^a and u^a , which, following [8], we will refer to as ‘*sheets*’. In the (1+3) approach any second rank symmetric 4-tensor can be split into a scalar along u^a , a 3-vector orthogonal to u^a , and a *projected symmetric trace free* (PSTF) 3-tensor. In (1+1+2) slicing, we take this split further by splitting the 3-vector and PSTF 3-tensors with respect to e^a . Any 3-vector, ψ^a , can be irreducibly split into a component along e^a and a sheet component Ψ^a orthogonal to e^a i.e. $\psi^a = \Psi e^a + \Psi^a$, where we have $\Psi \equiv \psi^a e_a$ and $\Psi^a \equiv N^{ab} \psi_b$. A similar decomposition can be done for PSTF 3-tensor, ψ_{ab} , which can be split into scalar (along e^a), 2-vector and PSTF 2-tensor.

The key variables of the 1+1+2 formalism obtained in this way are (see [23] for a detailed physical description of these variables)

$$[\Theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \phi, \xi, \mathcal{A}^a, \Omega^a, \Sigma^a, \alpha^a, a^a, \mathcal{E}^a, \mathcal{H}^a, \Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}, \zeta_{ab}] . \quad (5)$$

These variables (scalars, 2-vectors and PSTF 2-tensors) form an *irreducible set* and completely describe a vacuum spacetime.

In 1+3 formalism, the vector u^a is used to define the *covariant time derivative* (denoted by a dot) for any tensor $T^{a..b}_{c..d}$ along the observers’ worldlines defined by

$$\dot{T}^{a..b}_{c..d} = u^e \nabla_e T^{a..b}_{c..d} , \quad (6)$$

and the tensor h_{ab} is used to define the fully orthogonally *projected covariant derivative* D for any tensor $T^{a..b}_{c..d}$,

$$D_e T^{a..b}_{c..d} = h^a_f h^p_c \dots h^b_g h^q_d h^r_e \nabla_r T^{f..g}_{p..q} , \quad (7)$$

with total projection on all the free indices.

In 1+1+2 formalism, apart from the ‘*time*’ (dot) derivative of an object (scalar, vector or tensor) which is the derivative along the timelike congruence u^a , we now introduce two new derivatives, which e^a defines, for any object $\psi_{a..b}^{c..d}$:

$$\hat{\psi}_{a..b}^{c..d} \equiv e^f D_f \psi_{a..b}^{c..d} , \quad (8)$$

$$\delta_f \psi_{a..b}^{c..d} \equiv N_a^f \dots N_b^g N_h^c \dots N_i^d N_j^g D_j \psi_{f..g}^{i..j} . \quad (9)$$

The hat-derivative is the derivative along the e^a vector-field in the surfaces orthogonal to u^a . The δ -derivative is the projected derivative onto the orthogonal 2-sheet, with the projection on every free index.

3.2 Birkhoff Theorem for vacuum LRS-II spacetime

Locally Rotationally Symmetric (LRS) spacetimes [17] exhibit locally (at each point) a unique preferred spatial direction, which is covariantly defined [42]; the spacetime is invariant under rotations about this direction. Thus this symmetry requires the vanishing

of *all* orthogonal 1+1+2 vectors and tensors, such that there are no preferred directions in the sheet. Then, all the non-zero 1+1+2 variables are covariantly defined scalars.

A subclass of the LRS spacetimes, called LRS-II, contains all the LRS spacetimes that are rotation free; this is the class including the Schwarzschild solution. As a consequence, in LRS-II spacetimes the variables $\{\mathcal{A}, \Theta, \phi, \Sigma, \mathcal{E}\}$ fully characterize the kinematics (see [23] for the field equations governing their evolution and propagation). From the field equations of vacuum LRS-II spacetimes we get a very interesting result: the 1+1+2 scalar of the electric part of the Weyl tensor is always proportional to (3/2)th power of the Gaussian curvature of the 2-sheet. The proportionality constant sets up a scale in the problem.

To covariantly investigate the geometry of the vacuum LRS-II spacetime, let us try to solve the *Killing equation* for a Killing vector of the form $\xi_a = \Psi u_a + \Phi e_a$, where Ψ and Φ are scalars. The Killing equation is $\nabla_{(a}\xi_{b)} = 0$, from which we get the following differential equations and constraints:

$$\dot{\Psi} + \mathcal{A}\Phi = 0 \quad , \quad \hat{\Psi} - \dot{\Phi} - \Psi\mathcal{A} + \Phi(\Sigma + \frac{1}{3}\Theta) = 0, \quad (10)$$

$$\hat{\Phi} + \Psi(\frac{1}{3}\Theta + \Sigma) = 0 \quad , \quad \Psi(\frac{2}{3}\Theta - \Sigma) + \Phi\phi = 0. \quad (11)$$

Now we know $\xi_a \xi^a = -\Psi^2 + \Phi^2$. If ξ^a is timelike (that is $\xi_a \xi^a < 0$), then because of the arbitrariness in choosing the vector u^a , we can always make $\Phi = 0$. On the other hand, if ξ^a is spacelike (that is $\xi_a \xi^a > 0$), we can make $\Psi = 0$.

Let us first assume that ξ^a is timelike and $\Phi = 0$. In that case we know that the solution of equations (10) always exists while the constraints (11) imply that in general, (for a non trivial Ψ), $\Theta = \Sigma = 0$. Thus the expansion and shear of a unit vector field along the timelike Killing vector direction vanishes. In this case the spacetime is static. Now if ξ^a is spacelike and $\Psi = 0$, we always have a solution for Φ and $\phi = \mathcal{A} = 0$. Then from the field equations we can deduce that the spatial derivatives of all quantity vanish and hence the spacetime is spatially homogeneous. In other words, we can say: *There always exists a Killing vector in the local $[u, e]$ plane for a vacuum LRS-II spacetime. If the Killing vector is timelike then the spacetime is locally static, and if the Killing vector is spacelike the spacetime is locally spatially homogeneous.*

If the Gaussian curvature of the sheet is positive, then in the first case, we can solve the field equations to get a Schwarzschild exterior ($r > 2m$) vacuum metric, while in the second case we get a Schwarzschild interior ($r < 2m$) vacuum metric. Thus we have proved the (local)

Birkhoff Theorem: *Any C^2 solution of Einstein's equations in empty space which is spherically symmetric in an open set \mathcal{S} is locally equivalent to part of maximally extended Schwarzschild solution in \mathcal{S} .*

It is interesting to note that the modulus of the proportionality constant relating the 1+1+2 scalar of the electric part of the Weyl tensor and the Gaussian curvature is exactly equal to the Schwarzschild radius.

3.3 Almost Spherical Symmetry

To define the notion of an *almost spherically symmetric* spacetime, let us recall that the two dimensional Riemann curvature tensor of a 2-sheet can be written in terms of the Gaussian curvature ‘ K ’ as

$$^{(2)}R_{bcd}^a = K (N_c^a N_{bd} - N_d^a N_{bc}) . \quad (12)$$

Using the above equation we can immediately write the *geodesic deviation equation* for a family of closely spaced geodesics on the 2-sheets with tangent vectors $\psi^a(v)$ and separation vectors $\eta^a(v)$ (where ‘ v ’ is the parameter which labels the different geodesics) as

$$\psi^e \delta_e (\psi^f \delta_f \eta^a) = K (\psi^a \psi_d \eta^d - \eta^a \psi_c \psi^c) . \quad (13)$$

Let us now define a vector

$$Z^a = \psi^e \delta_e (\psi^f \delta_f \eta^a) - K_0 (\psi^a \psi_d \eta^d - \eta^a \psi_c \psi^c) . \quad (14)$$

Here K_0 is the Gaussian curvature for a spherical sheet (which is constant for a given value of affine parameter along the integral curves of the vector e^a). If the 2-sheets are exactly spherical then this vector vanishes and hence the magnitude of Z^a ($=\sqrt{Z_a Z^a}$) gives a covariant measure of the deviation from the spherical symmetry.

We now define an *almost spherically symmetric* spacetime in the following way:

Any C^2 spacetime with positive Gaussian curvature everywhere, which admits a local 1+1+2 splitting at every point is called an almost spherically symmetric spacetime, iff the following quantities are either zero or much smaller than the scale defined by the modulus of the proportionality constant (that relates the 1+1+2 Weyl scalar and the (3/2)th power of the Gaussian curvature).

- *The magnitude of all the 2-vectors (defined by $\sqrt{\psi_a \psi^a}$) and PSTF 2-tensors (defined by $\sqrt{\psi_{ab} \psi^{ab}}$) described in equation (5).*
- *The magnitude of the vector Z^a defined above in (14).*

We would like to emphasize here that though Minkowski spacetime belongs to the set of LRS-II, in the above definition of the perturbed spacetime we exclude the Minkowski background, as in that case the scale is identically zero.

3.4 Almost Birkhoff theorem for almost spherical symmetry

The set of all 1+1+2 variables in (5) apart from $\{\mathcal{A}, \Theta, \phi, \Sigma, \mathcal{E}\}$ are all of $\mathcal{O}(\epsilon)$ with respect to the invariant scale. Using equations (48-81) of [7], we can get the propagation and evolution equations of these small quantities.

In these equations all the zeroth order quantities are background quantities. If the background is static with $\Theta = \Sigma = 0$ and the time derivative all the background quantities are zero, the time derivatives of the first order quantities at a given point is of the same order of smallness as themselves. Hence the first order quantities still remains “small” as

the time evolves. Similarly if the background is spatially homogeneous with $\phi = \mathcal{A} = 0$ and the spatial derivative all the background quantities are zero, the spatial derivatives of the first order quantities at a given point are of the same order of smallness as themselves. Hence the first order quantities still remain “small” along the spatial direction. In both the cases of a static background and a spatially homogeneous background the resultant set of equations are the perturbed LRS-II equations.

Again trying to solve the Killing equation for a Killing vector of the form $\xi_a = \Psi u_a + \Phi e_a$, we get the following extra differential equations and constraints (apart from (10) and (11)):

$$-\delta_c \Psi + \Psi \mathcal{A}_c + \Phi \{ \varepsilon_{cd} \Omega^d + \alpha_c + \Sigma_c \} = 0 , \quad (15)$$

$$\delta_c \Phi + \Phi a_c + 2\Psi \Sigma_c = 0 , \quad \Psi \Sigma_{cd} + \Phi \zeta_{cd} = 0 . \quad (16)$$

Now we see that for both timelike ($\Phi = 0$) or spacelike ($\Psi = 0$) vectors, all the above equations are not completely solved in general for the arbitrary small values of the first order quantities. However as we proved that these first order quantities generically remain $\mathcal{O}(\epsilon)$ both in space and time, we can see that a timelike vector with ($\Theta = \Sigma = 0$) or a spacelike vector with ($\phi = \mathcal{A} = 0$) almost solves the Killing equations. Therefore we can say: *For an almost spherically symmetric vacuum spacetime there always exists a vector in the local $[u, e]$ plane which almost solves the Killing equations. If this vector is timelike then the spacetime is locally almost static, and if the Killing vector is spacelike the spacetime is locally almost spatially homogeneous.*

Also as we have seen that in this case the resultant set of equations are the perturbed LRS-II equations, with $\mathcal{O}(\epsilon)$ terms added to each, and the perturbations locally remain small both in space and time, a part of the maximally extended almost-Schwarzschild solution will then solve the field equations locally. Thus we have proved the (local)

Almost Birkhoff Theorem: *Any C^2 solution of Einstein’s vacuum equations which is almost spherically symmetric in an open set \mathcal{S} , is locally almost equivalent to part of a maximally extended Schwarzschild solution in \mathcal{S} .*

Note that we do not consider perturbations across the horizon: our result holds for any open set \mathcal{S} that does not intersect the horizon in the background spacetime. The result almost certainly holds true across the horizon also, but that case needs separate consideration. The above result can be immediately generalized in the presence of a cosmological constant. In that case an ‘almost’ spherically symmetric solution in an open set \mathcal{S} , is locally almost equivalent to part of a maximally extended Schwarzschild deSitter/anti-deSitter solution in \mathcal{S} . Thus the 2-mass solution above is stable to inhomogeneous perturbations in this sense.

What are the implications? First, the role of Birkhoff’s Theorem in astrophysics, characterizing the gravitational field in the vicinity of massive objects, is unchanged due to geometric perturbations: for example those due to Jupiter in the vicinity of the Sun or stellar rotation, which has considerable effects in perturbing spherical symmetry near the vicinity of the stars. In other words, the rigidity of spherical vacuum manifolds in General relativity continues even in the perturbed scenario. Secondly, this result is

unlikely to play any role as regards the expansion of the universe : such perturbations probably don't affect the way local static domains add up to give an expanding universe (for example rotational effects are stationary and do not help in explaining expansion).

4 Birkhoff with matter: finite infinity

We know that real astronomical systems are neither exactly spherically symmetric, nor exactly empty. While the Birkhoff theorem and its generalization remains valid for the case of an electrovac solution ([13], section 18.1), Birkhoff's theorem is not true in general when matter is present, as is shown for example by the Lemaître-Tolman-Bondi solutions [4, 31]. It remains true if the matter is static ([5], Section 4.3) but this will not be true in general. These results do not include crucial cases such as the Solar System, which is neither exactly empty nor exactly spherically symmetric.

In a previous section we showed that the result is stable to small geometric perturbations: it remains true if spacetime is not exactly spherically symmetric. Here we summarize [24], which shows that the result is stable to small matter perturbations: it remains true if spacetime is not exactly vacuum, as for example in the case of the solar system.

In other words, we would like to ask the question, how much matter can be present if the Birkhoff theorem is to remain approximately true. That is, we would like to perturb a vacuum LRS-II spacetime by introducing a small amount of general matter in the spacetime. In this section we only deal with the static exterior background as that is astrophysically more interesting case.

4.1 Matter

In the 1+3 splitting, the Energy Momentum Tensor T_{ab} of a general matter field can be written as

$$T_{ab} = \mu u_a u_b + q_a u_b + u_a q_b + p h_{ab} + \pi_{ab} \quad (17)$$

Where the scalars $\mu = T_{ab} u^a u^b$ and $p = (1/3) T_{ab} h^{ab}$ are the energy density and isotropic pressure respectively. The 3-vector, $q^a = T_{cb} u^b h^{ca}$, is the heat flux and the PSTF 3-tensor, $\pi_{ab} = T_{cd} h_{<a}^c h_{b>}^d$, defines the anisotropic stress. In 1+1+2 splitting of LRS-II spacetime, we can write the fluid variables as $q^a = Q e^a$ and $\pi_{ab} = \Pi [e_a e_b - \frac{1}{2} N_{ab}]$, where Q and Π are scalars.

We know from the covariant linear perturbation theory, any quantity which is zero in the background is considered as a first order quantity and is automatically gauge-invariant by virtue of the Stewart and Walker lemma [40]. Hence the set $\{\Theta, \Sigma, \mu, p, \Pi, Q\}$, describes the first order quantities, on a vacuum LRS-II background. As we have already seen, the vacuum spacetime has a covariant scale given by the Schwarzschild radius which sets up the scale for perturbation. Let us locally introduce general matter on a static

Schwarzschild background such that

$$\left[\frac{\mu}{K^{(3/2)}}, \frac{|p|}{K^{(3/2)}}, \frac{|\Pi|}{K^{(3/2)}}, \frac{|Q|}{K^{(3/2)}} \right] \ll C, \quad (18)$$

and

$$\left[\frac{|\hat{\mu}|}{K^{(3/2)}}, \frac{|\hat{p}|}{K^{(3/2)}}, \frac{|\hat{\Pi}|}{K^{(3/2)}}, \frac{|\hat{Q}|}{K^{(3/2)}} \right] \ll \phi C \quad (19)$$

where C is the proportionality constant described in the previous section (that relates the 1+1+2 Weyl scalar and the (3/2)th power of the local Gaussian curvature), which is also the Schwarzschild radius, and K is the local Gaussian curvature.

4.2 Domains

Now we need to make clear in what domain these equations will hold. The application will be to the spherically symmetric exterior domain of a star of mass M and Schwarzschild radius $R_S = 2M$, in the units of $8\pi G = c = 1$. We will define *Finite Infinity* \mathcal{F} as a 2-sphere of radius $R_{\mathcal{F}} \gg R_M$ surrounding the star: this is infinity for all practical purposes [19, 21]. We assume the relations (18, 19) hold in the domain $D_{\mathcal{F}}$ defined by $r_S < r < R_{\mathcal{F}}$ where $r_S > r_M$ is the radius of the surface of the star. This is the local domain where our results will apply. In the case of the solar system, $R_{\mathcal{F}}$ can be taken to be about a light year .

It is important to make this restriction, else eventually we will reach a radius r where these inequalities will no longer hold; in the real universe asymptotically flat regions are always of finite size, being replaced at larger scales by galactic and cosmological conditions. The result we wish to prove is a local result, applicable to the locally restricted nature of real physical systems.

4.3 Matter perturbations on vacuum LRS-II spacetimes

From the linearised matter conservation equations of the system (see [24] for detailed description of these equations) we can see that if (18) and (19) are locally satisfied at any epoch, within the domain $D_{\mathcal{F}}$, then the time variation of the matter variables are of same order of smallness as themselves. Hence there exists an open set \mathcal{S} within where the amount of matter remains “*small*”, if the amount is small at any epoch in \mathcal{S} and only small amounts of matter enter $D_{\mathcal{F}}$ across \mathcal{F} .

One could attempt to determine the same kinds of inequality as those above for matter crossing \mathcal{F} , but one can resolve this issue in another way: we have not yet specified the time evolution of \mathcal{F} . We now do so in the following manner: choose it in a suitable manner in some initial surface $t = t_0$, and then propagate it to the future by dragging it along world lines that are integral curves of the timelike eigenvector of the Ricci tensor R_{ab} (this will be unique for any realistic non-zero matter). As these are then timelike eigenvectors of the stress tensor T_{ab} (because of the field equations), equal amount of energy density will convect in and out across \mathcal{F} due to random motions of matter [14]; the total amount of matter inside \mathcal{F} will be conserved, and if the inequalities (18, 19) are satisfied at some

initial time they will be satisfied at later times, unless major masses enter the \mathcal{F} locally in some region. If this is so, we do not have an isolated system and the extended Birkhoff's theorem need not apply.

Hence we will define the time evolution of \mathcal{F} in the way just indicated, and suppose that (18, 19) are then satisfied at later times; if this is not the case the local system considered is not isolated and our result is not applicable.

4.4 Almost symmetries

From the linearised Field equations, it is evident that if the matter variables remain “small” as defined by (18) then the spatial and temporal variance of the expansion Θ and the shear Σ are of the same order of smallness as the matter. In that case we see that a timelike vector will not exactly solve the Killing equations (10)-(11) in general, although it may do so approximately.

To see this explicitly, let us set $\Phi = 0$ in Killing's equation and consider the following symmetric tensor $K_{ab} := \nabla_a(\Psi u_b) + \nabla_b(\Psi u_a)$. This tensor vanishes if Ψu^a is a Killing vector. This is the case of an exact symmetry when the spacetime is exactly static. However, in the perturbed scenario, to see how close the vector $\xi_a = \Psi u_a$ is to a Killing vector, let us consider the scalars constructed by contracting the above tensor by the vectors u^a , e^a and the projection tensor N^{ab} . If the conditions

$$\left[\frac{|K_{ab}u^a u^b|^2}{K^{3/2}}, \frac{|K_{ab}u^a e^b|^2}{K^{3/2}}, \frac{|K_{ab}e^a e^b|^2}{K^{3/2}}, \frac{|K_{ab}N^{ab}|^2}{K^{3/2}} \right] \ll C \quad (20)$$

are satisfied, then we can say that $\xi_a = \Psi u_a$ is close to a Killing vector and the spacetime is approximately static. From the previous section (using equation (10)) we know that there always exist a non-trivial solution of the scalar Ψ for which $|K_{ab}u^a u^b|$ and $|K_{ab}u^a e^b|$ vanishes; we choose Ψ accordingly. However for a general matter perturbation, as Θ and Σ are non-zero, $|K_{ab}e^a e^b|^2$ and $|K_{ab}N^{ab}|^2$ are generally non-zero. However, using the linearised field equations we get

$$\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma \right)^2 \approx \frac{1}{3}\mu - \frac{1}{2}\Pi. ; \quad \left(\frac{1}{3}\Theta + \Sigma \right) \left(\frac{2}{3}\Theta - \Sigma \right) \approx \frac{2}{3}\mu + \frac{1}{2}\Pi. \quad (21)$$

Thus we see that if the amount of matter is “small”, that is the condition (18) is satisfied, then the following conditions are satisfied

$$|K_{ab}e^a e^b|^2 = \Psi^2 \left(\frac{1}{3}\Theta + \Sigma \right)^2 \ll CK^{3/2}, \quad (22)$$

$$|K_{ab}N^{ab}|^2 = \Psi^2 \left(\frac{2}{3}\Theta - \Sigma \right)^2 \ll CK^{3/2}. \quad (23)$$

Therefore we can say that there always exists a timelike vector that satisfies (20). This vector then almost solves the Killing equations in \mathcal{S} and hence the spacetime is *almost* static in \mathcal{S} . This is the **Almost Birkhoff theorem** for an almost vacuum spherically symmetric solution.

The above conditions, (18) and (19), can also be written in another way.

$$\left[\frac{|R|}{K^{(3/2)}}, \frac{|R_{ab}u^a u^b|}{K^{(3/2)}}, \frac{|R_{<ab>}e^a e^b|}{K^{(3/2)}}, \frac{|R_{<ab>}u^a e^b|}{K^{(3/2)}} \right] << C \quad (24)$$

and

$$\left[\frac{|\hat{R}|}{K^{(3/2)}}, \frac{|R_{ab}u^a u^b|}{K^{(3/2)}}, \frac{|R_{<ab>}u^a e^b|}{K^{(3/2)}}, \frac{|R_{<ab>}e^a e^b|}{K^{(3/2)}} \right] << \phi C \quad (25)$$

In other words the ratio of the scalars constructed from the Ricci tensor using the vectors u^a and e^a (and their spatial variations) to the (3/2)th power of the local Gaussian curvature of the 2-sheet should be much smaller than the Schwarzschild radius if the Birkhoff theorem is to remain approximately true. Equations (24) and (25) are easier to use, in case of presence of multifluids in the spacetime.

What are the implications? First, the role of Birkhoff's Theorem in astrophysics, characterizing the gravitational field in the vicinity of massive objects, is unchanged due to small matter perturbations, for example dust or dark matter pervading the solar system. Secondly, the solution is almost but not exactly static, and this might indeed play a role as regards the expansion of the universe. Dark matter could conceivably affect the local static domains so they each give a small contribution to an expanding universe, which adds up to give the global effect we see. But then we should be able to measure the Hubble constant in the solar system, at least in principle (See Section 2.6),

4.5 Comments on the solar system

In case of the solar system [33] we know that the interplanetary medium includes interplanetary dust, cosmic rays and hot plasma from the solar wind. Its density is very low at about 5 particles per cubic centimeter in the vicinity of the Earth; it decreases with increasing distance from the sun, in inverse proportion to the square of the distance. In this section, to compare our result with the observed astronomical data, we will use SI units for clarity.

The density of interplanetary medium is variable, and may be affected by magnetic fields and events such as coronal mass ejections. It may rise to as high as 100 particles/cm³. These particles are mostly Hydrogen nuclei, and hence the maximum density per cubic meter will be approximately of the order of 10⁻¹⁹ Kilograms, and the local Gaussian curvature of the heliocentric celestial sphere in the vicinity of the earth is of the order of 10⁻²² m⁻². Hence the ratio of the maximum interplanetary density to the (3/2)th power of the local Gaussian curvature is of the order of 10¹⁴ Kilograms, which is much smaller than the solar mass (10³⁰ Kilograms). Also the large amplitude waves in the medium are comparable to the energy density of the unperturbed medium, which makes the spatial variation of energy density to be of the same order of smallness as itself. This satisfies (18) and (19) and hence in the solar system the Birkhoff theorem remains almost true.

We can relate the discussion to the *Finite Infinity* concept for the solar system. We know that the outer edge of the solar system is the boundary between the flow of the

solar wind and the diffused interstellar medium. This boundary, which is known as the *Heliopause*, is at a radius of approximately 10^{13} meters. The interplanetary medium thus fills the roughly spherical volume contained within the heliopause. As the density of the interplanetary medium decreases in inverse proportion to the square of the distance, the density near the heliopause is of the order of 10^{-23} Kilograms per cubic meter. Hence the ratio of the density to the $(3/2)$ th power of the local Gaussian curvature is of the order of 10^{16} Kilograms and still remains much smaller than the solar mass. Also the amount of matter crossing the heliopause to the diffused interstellar medium is of the same order. Hence we can easily define the heliopause as the boundary of our domain $D_{\mathcal{F}}$. As the conditions (18) and (19) are true at the boundary of the domain, they should be true everywhere inside the domain, unless the matter outside the star is highly clustered locally. But we are considering the case of a low density diffuse gas where this is not the case. the conditions (24) and (25) will be satisfied in this domain.

For the massive planets inside the solar system (e.g. Jupiter or Saturn), these conditions may be violated in their very close vicinity, but in that case the local spacetime no longer remains spherically symmetric. This can be easily checked by calculating the geodesic deviation equation near the Lagrangian points of these planets and calculating the magnitude of the vector Z^a described in the previous section. However as the vast fraction of the solar system's mass (more than 99%) is in the sun, on average these massive planets have a very tiny effect on the system as a whole and the approximate theorem remains true. Hence the local spacetime within the solar system is “*almost*” described by a Schwarzschild metric. Note that we have not included dark matter in this mass budget. We are unaware of any claims that it is significant in the solar system.

5 The expansion of the universe

Birkhoff's theorem is a key theorem relating the global expansion of the universe to local gravitating systems: it protects them from the expansion of the universe, and raises interesting issues as to how such static domains can be patched together to give an expanding universe.

We have explored these relations here to see how they can be compatible, inter alia summarising two useful generalisations of Birkhoff's theorem. The point that arises is whether dark matter - the dominant form of matter in the universe at large scales - pervades the solar system and links our local system to the cosmic expansion. This seems unlikely – but if it is true, cosmology can in principle be done in the solar system by measuring its time-changing effects on solar system dynamics.

Acknowledgements: We thank Chris Clarkson for helpful comments, and the NRF and UCT Research Committee for financial support.

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